MINIMAL LENGTH ELEMENTS OF FINITE COXETER GROUPS

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Abstract. We give a geometric proof that minimal length elements in a (twisted) conjugacy class of a finite Coxeter group W have remarkable properties with respect to conjugation, taking powers in the associated Braid group and taking centralizer in W.

Introduction

Let W be a finite Coxeter group. Let $\mathcal{O}$ be a conjugacy class of W and $\mathcal{O}_{\min}$ be the set of elements of minimal length in $\mathcal{O}$. In [GP1], Geck and Pfeiffer showed that the elements in $\mathcal{O}_{\min}$ have remarkable properties with respect to conjugation in W and in the associated Hecke algebra H. In [GM], Geck and Michel showed that there exists some element in $\mathcal{O}_{\min}$ that has remarkable properties when taking powers in the associated Braid group. These properties were later generalized to twisted conjugacy classes. See [GKP] and [H1]. In a recent work [L3], Lusztig showed that the centralizer of a minimal length element in W also has remarkable properties.

These remarkable properties lead to the definition of determination of "character tables" of Hecke algebra H. They also play an important role in the study of unipotent representation [L2] and in the study of geometric and cohomological properties of Deligne-Lusztig varieties [OR], [H2], [BR], [L3] and [HL].

The proofs of these properties were of a case-by-case nature and replied on computer calculation for exceptional types.

In this paper, we'll give a case-free proof of these properties on $\mathcal{O}_{\min}$ based on a geometric interpretation of conjugacy classes and length function on W. Similar ideas will also be applied to affine Weyl groups in a future work.

In [R], Rapoport pointed out to us that this paper together with [OR], [BR] (see also [HL] for a stronger result) gives a computer-free proof of the vanishing theorem [OR, 2.1] on the cohomology of Deligne-Lusztig varieties. This simplifies several steps in Lusztig' classification of representation of finite groups of Lie type [L1]. More precisely, the

2000 *Mathematics Subject Classification.* 20F55, 20E45.

Key words and phrases. Minimal length element, good element, finite Coxeter group.

proof of [L1, Proposition 6.10] applies without the assumption of $q \geqslant h$ and many arguments in [L1, Chapter 7-9] can then be bypassed.

1. Geometric interpretation of twisted conjugacy class

1.1. Let W be a finite Coxeter group with generators s_i for $i \in S$ and corresponding Coxeter matrix $M = (m_{ij})_{i,j \in S}$. The elements s_i for $i \in S$ are called *simple reflections*. Let $\delta : W \to W$ be a group automorphism sending simple reflections to simple reflections. We still denote by δ the induced bijection on S. Then $\delta(s_i) = s_{\delta(i)}$ for all $i \in S$. Set $\tilde{W} = \langle \delta \rangle \ltimes W$. For any $i \in \mathbb{Z}$ and $w \in W$, we set $\ell(\delta^i w) = \ell(w)$, where $\ell(w)$ is the length of w in the Coxeter group W.

For any subset J of S, we denote by W_J the standard parabolic subgroup of W generated by $(s_i)_{i \in J}$ and by w_J the maximal element in W_J. We denote by $\tilde{W}^J$ (resp. ${}^J\tilde{W}$) the set of minimal coset representatives in $\tilde{W}/W_J$ (resp. $W_J \backslash \tilde{W}$). We simply write ${}^J\tilde{W}^J$ for ${}^J\tilde{W} \cap \tilde{W}^J$. For $\tilde{w} \in {}^J\tilde{W}^J$, we write $\tilde{w}(J) = J$ if the conjugation of $\tilde{w}$ sends simple reflections in J to simple reflections in J.

1.2. Two elements w, w' of W are said to be δ-twisted conjugate if $w' = \delta(x) w x^{-1}$ for some $x \in W$. The relation of δ-twisted conjugacy is an equivalence relation and the equivalence classes are said to be δ-twisted conjugacy classes.

For $w, w' \in W$ and $i \in S$, we write $w \xrightarrow{s_i}_\delta w'$ if $w' = s_{\delta(i)} w s_i$ and $\ell(w') \leqslant \ell(w)$. We write $w \to_\delta w'$ if there is a sequence $w = w_0, w_1, \cdots, w_n = w'$ of elements in W such that for any k, $w_{k-1} \xrightarrow{s_i}_\delta w_k$ for some $i \in S$.

We write $w \approx_\delta w'$ if $w \to_\delta w'$ and $w' \to_\delta w$. It is easy to see that $w \approx_\delta w'$ if $w \to_\delta w'$ and $\ell(w) = \ell(w')$.

We call $w, w' \in W$ *elementarily strongly δ-conjugate* if $\ell(w) = \ell(w')$ and there exists $x \in W$ such that $w' = \delta(x) w x^{-1}$ and $\ell(\delta(x) w) = \ell(x) + \ell(w)$ or $\ell(w x^{-1}) = \ell(x) + \ell(w)$. We call w, w' *strongly δ-conjugate* if there is a sequence $w = w_0, w_1, \cdots, w_n = w'$ such that for each i, w_{i-1} is elementarily strongly δ-conjugate to w_i. We write $w \sim_\delta w'$ if w and w' are strongly δ-conjugate.

Now we translate the notations $\to_\delta, \sim_\delta, \approx_\delta$ in W to some notations in $\tilde{W}$.

By [GKP, Remark 2.1], the map $w \mapsto \delta w$ gives a bijection between the δ-twisted conjugacy classes of W and the ordinary conjugacy classes of $\tilde{W}$ that is contained in $W\delta$.

For $\tilde{w}, \tilde{w}' \in \tilde{W}$ and $i \in S$, we write $\tilde{w} \xrightarrow{s_i} \tilde{w}'$ if $\tilde{w}' = s_i \tilde{w} s_i$ and $\ell(\tilde{w}') \leqslant \ell(\tilde{w})$. We write $\tilde{w} \to \tilde{w}'$ if there is a sequence $\tilde{w} = \tilde{w}_0, \tilde{w}_1, \cdots, \tilde{w}_n = \tilde{w}'$ of elements in $\tilde{W}$ such that for any k, $\tilde{w}_{k-1} \xrightarrow{s_i} \tilde{w}_k$ for some $i \in S$. The notations $\approx$ and $\sim$ on $\tilde{W}$ are also defined in a similar way as $\approx_\delta$, $\sim_\delta$

on W. Then it is easy to see that for any $w, w' \in W$, $w \to_\delta w'$ iff $\delta w \to \delta w'$, $w \approx_\delta w'$ iff $\delta w \approx \delta w'$ and $w \sim_\delta w'$ iff $\delta w \sim \delta w'$.

1.3. Let V be a real vector space with inner product $(,)$ such that there is an injection $\tilde{W} \hookrightarrow GL(V)$ preserving $(,)$ and for any $i \in S$, s_i acts on V as a reflection. By [B, Ch.V], such V always exists. Unless otherwise stated, we regard $\tilde{W}$ as a reflection subgroup of $GL(V)$. We denote by $||\cdot||$ the norm on V defined by $||v|| = \sqrt{(v,v)}$ for $v \in V$.

For any subspace U of V, we denote by $U^\perp$ its orthogonal complement.

For any hyperplane H, let $s_H \in GL(V)$ be the reflection along H. Let $\mathfrak{H}$ be the set of hyperplanes H of V such that the reflection s_H is in W. Let V^W be the set of fixed points by the action of W. Since W is generated by s_H for $H \in \mathfrak{H}$, $V^W = \cap_{H \in \mathfrak{H}} H$.

Even if we start with V with no nonzero fixed points, some pair (W', V') with $(V')^{W'} \neq \{0\}$ appears in the inductive argument in this paper. This is the reason that we consider some vector space other than the one introduced in [B, Ch.V].

A connected component A of $V - \cup_{H \in \mathfrak{H}} H$ is called a *Weyl chamber*. We denote its closure by $\bar{A}$. Let $H \in \mathfrak{H}$, if the set of inner points $H_A = (H \cap \bar{A})^\circ \subset H \cap \bar{A}$ spans H, then we call H a *wall* of A and H_A a *face* of A.

The Coxeter group W acts simply transitively on the set of Weyl chambers. The chamber containing $C = \{x \in E; (x, e_i) > 0 \text{ for all } i \in S\}$ is called the *fundamental chamber* which is also denoted by C. For any Weyl chamber A, we denote by x_A the unique element in W such that $x_A(C) = A$.

Let $K \subset V$ be a convex subset. A point $x \in K$ is called *a regular point* of K if for each $H \in \mathfrak{H}$, $K \subset H$ whenever $x \in H$. The set of regular points of K is open dense in K.

1.4. Given any element $\tilde{w} \in \tilde{W}$ and a Weyl chamber A, we define $\tilde{w}_A = x_A^{-1} \tilde{w} x_A$. Then the map $A \mapsto \tilde{w}_A$ gives a bijection from the set of Weyl chambers to the conjugacy class of $\tilde{w}$ in $\tilde{W}$.

For any two chambers A, A', denote by $\mathfrak{H}(A, A')$ the set of hyperplanes in $\mathfrak{H}$ separating A and A'. Then $\ell(\tilde{w}) = \sharp \mathfrak{H}(C, \tilde{w}(C))$ for any $\tilde{w} \in \tilde{W}$. In general for any Weyl chamber A,

$$\ell(\tilde{w}_A) = \sharp \mathfrak{H}(A, \tilde{w}(A)).$$

Let A, A' be Weyl Chambers with a common face $H_A = H_{A'}$, here $H \in \mathfrak{H}$. Then $x_A^{-1} s_H x_A = s_i$ for some $i \in S$. Now

$$\tilde{w}_{A'} = (s_H x_A)^{-1} \tilde{w} (s_H x_A) = s_i x_A^{-1} \tilde{w} x_A s_i = s_i \tilde{w}_A s_i$$

is obtained from $\tilde{w}_A$ by conjugation a simple reflection s_i. We may check if $\ell(\tilde{w}_{A'}) > \ell(\tilde{w}_A)$ by the following criterion.

Lemma 1.1. *We keep the notations as above. Define a map* $f_{\tilde{w}} : V \to \mathbb{R}$ *by* $v \mapsto ||\tilde{w}(v) - v||^2$. *Let* $h \in H_A$ *and* $v \in H^{\perp}$ *with* $x - \epsilon v \in A$ *for sufficiently small* $\epsilon > 0$. *Set* $D_v f(h) = \lim_{t \to 0} \frac{f_{\tilde{w}}(h+tv) - f_{\tilde{w}}(h)}{t}$. *If* $\ell(\tilde{w}_{A'}) = \ell(\tilde{w}_A) + 2$, *then* $D_v f(h) > 0$.

Proof. It is easy to see that $\mathfrak{H}(A', \tilde{w}A') - \mathfrak{H}(A, \tilde{w}A) \subset \{H, \tilde{w}H\}$. By our assumption $\sharp\mathfrak{H}(A', \tilde{w}A') = \sharp\mathfrak{H}(A, \tilde{w}A) + 2$. Hence

$$\mathfrak{H}(A', \tilde{w}A') = \mathfrak{H}(A, \tilde{w}A) \sqcup \{H, \tilde{w}H\}$$

and $H \neq \tilde{w}H$. In particular, $H_A \cap \tilde{w}H_A = \emptyset$ and $h \neq \tilde{w}(h)$.

Let $L(h, \tilde{w}(h))$ be the affine line spanned by h and $\tilde{w}(h)$. Then $L(h, \tilde{w}(h)) - H \cup \tilde{w}(H)$ consists of three connected components: $L_- = \{h + t(\tilde{w}(h) - h); t < 0\}$, $L_0 = \{h + t(\tilde{w}(h) - h); 0 < t < 1\}$ and $L_+ = \{h + t(\tilde{w}(h) - h); t > 0\}$. Note that $\mathfrak{H}(A, \tilde{w}A) \cap \{H, \tilde{w}H\} = \emptyset$, $A \cap L_0$ and $\tilde{w}A \cap L_0$ are nonempty. Since $(v, H) = 0$ and $h + v, h + (h - \tilde{w}(h))$ are in the same component of $V - H$, we have $(v, h - \tilde{w}(h)) > 0$. Similarly we have $(\tilde{w}(v), \tilde{w}(h) - h) > 0$. Now

$$\begin{aligned} \mathrm{D}_v f(h) &= 2(\tilde{w}(h) - h, \tilde{w}(v) - v) \\ &= 2(\tilde{w}(h) - h, \tilde{w}(v)) + 2(h - \tilde{w}(h), v) > 0. \end{aligned}$$

□

1.5. Let $\mathrm{grad} f_{\tilde{w}}$ be the gradient of $f_{\tilde{w}}$ on V, that is, for any vector field X on V, $X f_{\tilde{w}} = (X, \mathrm{grad} f_{\tilde{w}})$. Here we naturally identify V with the tangent space of any point in V. Then it is easy to see that $\mathrm{grad} f_{\tilde{w}}(v) = 2(1 - {}^t\tilde{w})(1 - \tilde{w})v$, where ${}^t\tilde{w}$ is the transpose of $\tilde{w}$ with respect to $(,)$. Let $C_{\tilde{w}} : V \times \mathbb{R} \to V$ be the integral curve of $\mathrm{grad} f_{\tilde{w}}$ with $C_{\tilde{w}}(v, 0) = v$ for all $v \in V$. Then

$$C_{\tilde{w}}(v, t) = \exp(2t(1 - {}^t\tilde{w})(1 - \tilde{w}))v.$$

Let $S(V) = \{v \in V; (v, v) = 1\}$ be the unit sphere of V. For any $0 \neq v \in V$, set $\overline{v} = \frac{v}{||v||} \in S(V)$. Define $p : V - \{0\} \to S(V)$ by $v \mapsto \lim_{t \to -\infty} \overline{C_{\tilde{w}}(v, t)}$.

In order to study the map p, we need to understand the eigenspace of $\tilde{w}$ on V.

1.6. Let $\tilde{w} \in \tilde{W}$. Let $\Gamma_{\tilde{w}}$ be the set of elements $\theta \in [0, \pi]$ such that $e^{i\theta}$ is an eigenvalue of $\tilde{w}$ on V.

For $\theta \in \Gamma_{\tilde{w}}$, we define

$$V_{\tilde{w}}^{\theta} = \{v \in V; \tilde{w}(v) + \tilde{w}^{-1}(v) = 2\cos\theta v\}.$$

Then $V_{\tilde{w}} \otimes_{\mathbb{R}} \mathbb{C}$ is the sum of eigenspaces of $V \otimes_{\mathbb{R}} \mathbb{C}$ with eigenvalues $e^{\pm i\theta}$. In particular, if θ is not 0 or π, then $V_{\tilde{w}}^{\theta}$ is an even-dimensional subspace of V over $\mathbb{R}$ on which $\tilde{w}$ acts as a rotation by θ.

Since $\tilde{w}$ is a linear isometry of finite order, we have an orthogonal decomposition

$$V = \bigoplus_{\theta \in \Gamma_w} V_{\tilde{w}}^{\theta}.$$

Let θ_0 be the minimal element in $\Gamma_{\tilde{w}}$ with $V_{\tilde{w}}^{\theta} \neq V^W$ and $V_{\tilde{w}} = V_{\tilde{w}}^{\theta_0} \cap (V^W)^{\perp}$.

Now for any $v_\theta \in V_{\tilde{w}}^{\theta}$,

$$\begin{aligned}(1 - {}^t\tilde{w})(1 - \tilde{w})v_\theta = (1 - e^{i\theta})(1 - e^{-i\theta})v_\theta &= ((1 - \cos\theta)^2 + \sin^2\theta)v_\theta \\ &= 2(1 - \cos\theta)v_\theta.\end{aligned}$$

In particular, let $v \in (V^W)^{\perp}$, then $v = \sum v_\theta$, where $v_\theta \in V_{\tilde{w}}^{\theta}$ and the summation is over all $\theta \in \Gamma_{\tilde{w}}$ with $\theta \geqslant \theta_0$. Then $C_{\tilde{w}}(v, t) = \sum \exp(4t(1 - \cos\theta))v_\theta$ and $p(v) = \overline{v}_{\theta_0}$ whenever $v_{\theta_0} \neq 0$.

Hence $p((V^W)^{\perp} - V_{\tilde{w}}^{\perp}) = S(V_{\tilde{w}})$ and $p : (V^W)^{\perp} - V_{\tilde{w}}^{\perp} \to S(V_{\tilde{w}})$ is a fiber bundle.

Proposition 1.2. *Let $\tilde{w} \in \tilde{W}$ and A be a Weyl chamber. Then there exists a Weyl Chamber A' such that $\bar{A}'$ contains a regular point of $V_{\tilde{w}}$ and $\tilde{w}_A \to \tilde{w}_{A'}$.*

Proof. Let $V_{\tilde{w}}^{\geqslant 1} \subset V_{\tilde{w}}$ be the complement of the set of regular points of $V_{\tilde{w}}$. By §1.6, $p^{-1}(V_{\tilde{w}}^{\geqslant 1})$ is a finite union of submanifolds of codimension $\geqslant 1$. Let $V^{\geqslant 2}$ be the complement of all chambers and faces in V, that is, the skeleton of V of codimension $\geqslant 2$. Then $C_{\tilde{w}}(V^{\geqslant 2}, \mathbb{R}) \subset V$ is a countable union of images, under smooth maps, of manifolds with dimension at most $\dim V - 1$. Let

$$D_{\tilde{w}} = \{v \in V; v \notin C_{\tilde{w}}(V^{\geqslant 2}, \mathbb{R}) \cup p^{-1}(V_{\tilde{w}}^{\geqslant 1}) \cup V_{\tilde{w}}^{\perp}\}.$$

Then $D_{\tilde{w}}$ is a dense subset in the sense of Lebesgue measure.

Choose $y \in A \cap D_{\tilde{w}}$. Set $x = p(y) \in V_{\tilde{w}}$. Then x is a regular point in $V_{\tilde{w}}$. There exists $T > 0$ such that for any chamber B, $x \in \bar{B}$ whenever $C_{\tilde{w}}(y, -T) \in \bar{B}$.

Now we define A_i, H_i, h_i, t_i as follows.

Set $A_0 = A$. Suppose A_i is defined and $A_i \neq A'$, then we set $t_i = \sup\{t < T; C_{\tilde{w}}(y, -t) \in \bar{A}_i\}$. Then $t_i \leqslant T$. Set $h_i = C_{\tilde{w}}(y, -t_i)$. By the definition of $D_{\tilde{w}}$, h_i is contained in a unique face of A_i, which we denote by H_i. Let $A_{i+1} \neq A_i$ be the unique chamber such that H_i is a common face of A_i and A_{i+1}. Then $C_{\tilde{w}}(y, -t_i - \epsilon) \in A_{i+1}$ for sufficiently small $\epsilon > 0$.

Since the chambers appear in the above list are distinct with each other. Thus the above procedure stops after finitely many steps. We obtain a finite sequence of chambers $A = A_0, A_1, \cdots, A_r = A'$ in this way. Since $C_{\tilde{w}}(y, -T) \in \bar{A}'$, we have $x \in \bar{A}'$.

Let $v_i \in V$ such that $(v_i, h_i - h) = 0$ for $h \in H_i$ and $h_i - \epsilon v_i \in A_i$ for sufficiently small $\epsilon > 0$. Since $C_{\tilde{w}}(y, -t_i - \epsilon') \in A_{i+1}$ for sufficiently

small $\epsilon' > 0$, $\mathrm{D}_{v_i} f_{\tilde{w}}(h_i) = (v_i, (\mathrm{grad} f_{\tilde{w}})(h_i)) \leqslant 0$. Hence by Lemma 1.1, $\ell(\tilde{w}_{A_{i+1}}) \leqslant \ell(\tilde{w}_{A_i})$ and $\tilde{w}_{A_i} \to \tilde{w}_{A_{i+1}}$.

Therefore $\tilde{w}_A \to \tilde{w}_{A'}$ and $\bar{A}'$ contains a regular point x of $V_{\tilde{w}}$. $\square$

2. Length formula

2.1. The main goal of this section is to give a length formula for the element $\tilde{w}_A$ with $\bar{A}$ containing a regular point of some subspace of V preserved by $\tilde{w}$.

Let $K \subset V$ be a convex subset. Let $\mathfrak{H}_K = \{H \in \mathfrak{H}; K \subset H\}$ and $W_K \subset W$ be the subgroup generated by s_H ($H \in \mathfrak{H}_K$). For any two chambers A and A', set $\mathfrak{H}(A, A')_K = \mathfrak{H}(A, A') \cap \mathfrak{H}_K$.

Let A be a Weyl chamber. We set $W_{K,A} = x_A^{-1} W_K x_A$. If $\bar{A}$ contains a regular point k of K, then we set $W_{K,A} = W_{I(K,A)}$ is the parabolic subgroup of W generated by simple reflections $I(K, A) = \{s_H \in S; k \in x_A H\}$.

It is easy to see that $\bar{A}$ contains a regular point of K if and only if it contains a regular point of $K + V^W$. In this case, $I(K, A) = I(K + V^W, A)$.

If A' is a Weyl chamber such that $\bar{A}'$ also contains k. Then there exists $x \in W_K$ with $x(A) = A'$. We set $x_{A,A'} = x_A^{-1} x x_A$. Then $x_{A,A'} \in W_{K,A}$ and

$$\tilde{w}_{A'} = (x x_A)^{-1} \tilde{w} (x x_A) = (x_A x_{A,A'})^{-1} \tilde{w} (x_A x_{A,A'}) = x_{A,A'}^{-1} \tilde{w}_A x_{A,A'}.$$

Moreover,

$$\ell(x_{A,A'}) = \sharp \mathfrak{H}(C, x_{A,A'}(C)) = \sharp \mathfrak{H}(x_A(C), x x_A(C)) = \sharp \mathfrak{H}(A, A').$$

We first consider the follows special case.

Lemma 2.1. *Let $\tilde{w} \in \tilde{W}$ and $K \subset V_{\tilde{w}}^{\theta}$ be a subspace such that $\tilde{w}(K) = K$. Let A be a Weyl chamber such that A and $\tilde{w}(A)$ are in the same connected component of $V - \cup_{H \in \mathfrak{H}_K} H$. Assume furthermore that $\bar{A}$ contains a nonzero element $v \in K$ such that for each $H \in \mathfrak{H}$, $v, \tilde{w}(v) \in H$ implies that $K \subset H$. Then*

$$\ell(\tilde{w}_A) = \sharp \mathfrak{H}(A, \tilde{w}(A)) = \frac{\theta}{\pi} \sharp (\mathfrak{H} - \mathfrak{H}_K).$$

Proof. By our assumption, $\mathfrak{H}(A, \tilde{w}(A)) \subset \mathfrak{H} - \mathfrak{H}_K$. Moreover, for any $H \in \mathfrak{H}(A, \tilde{w}(A))$, the intersection of H with the closed interval $[v, \tilde{w}(v)]$ is nonempty.

If $\theta = 0$, then $\tilde{w}(v) = v$. For $H \in \mathfrak{H}(A, \tilde{w}(A))$, $v \in H$ and hence $H \in \mathfrak{H}_K$. That is a contradiction. Hence $\mathfrak{H}(A, \tilde{w}(A)) = \emptyset$ and $\ell(\tilde{w}_A) = \sharp \mathfrak{H}(A, \tilde{w}(A)) = 0$.

If $\theta = \pi$, then $\tilde{w}(v) = -v$. We see $\mathfrak{H}(A, \tilde{w}(A)) = \mathfrak{H} - \mathfrak{H}_K$. Thus $\ell(\tilde{w}_A) = \sharp(\mathfrak{H} - \mathfrak{H}_K)$.

Now we assume $0 < \theta < \pi$ and d is the order of $\tilde{w}$. Set $v_i = \tilde{w}^i(\bar{v}) \in S(K)$ for $i \in \mathbb{Z}$. Since $\tilde{w}$ acts on K by rotation by θ, there exists a

2-dimensional subspace of K that contains v_i for all i. Let S^1 be the unit circle in this subspace. Let $Q_i \subset S^1$ be the open arc of angle θ connecting v_i with v_{i+1} and $Q_i' = Q_i \sqcup \{v_i\}$.

Let $H \in \mathfrak{H}(\tilde{w}^i(A), \tilde{w}^{i+1}(A))$. Then by our assumption, $H \in \mathfrak{H} - \mathfrak{H}_K$. If $v_i \notin H$ and $v_{i+1} \notin H$, then $H \cap Q_i \neq \emptyset$. On the other hand, for any $H \in \mathfrak{H}$, if $H \cap Q_i \neq \emptyset$, then $H \in \mathfrak{H}(\tilde{w}^i(A), \tilde{w}^{i+1}(A))$.

If $H \in \mathfrak{H} - \mathfrak{H}_K$ and $v_i \in H$, then $v_{i-1}, v_{i+1} \notin H$ and $\{v_i\}$ is the intersection of H with the open arc connecting v_{i-1} with v_{i+1} passing through v_i. Hence H belongs to exactly one of the two sets: $\mathfrak{H}(\tilde{w}^{i-1}(A), \tilde{w}^i(A))$ and $\mathfrak{H}(\tilde{w}^i(A), \tilde{w}^{i+1}(A))$. Therefore

$$(*) \qquad \sum_{i=0}^{d-1} \sharp\mathfrak{H}(\tilde{w}^i(A), \tilde{w}^{i+1}(A)) = \sum_{i=0}^{d-1} \sharp\{H \in \mathfrak{H} - \mathfrak{H}_K; H \cap Q_i' \neq \emptyset\}.$$

Notice that each $H \in \mathfrak{H} - \mathfrak{H}_K$ intersects S^1 at exactly 2 points. Hence H appears on the right hand side of $(*)$ exactly $d\theta/\pi$-times. Now

$$d\ell(\tilde{w}_A) = d\sharp\mathfrak{H}(A, \tilde{w}(A)) = \frac{d\theta}{\pi}\sharp(\mathfrak{H} - \mathfrak{H}_K)$$

and $\ell(\tilde{w}_A) = \frac{\theta}{\pi}\sharp(\mathfrak{H} - \mathfrak{H}_K)$. □

Proposition 2.2. *Let $\tilde{w} \in \tilde{W}$ and $K \subset V_{\tilde{w}}^{\theta}$ be a subspace with $\tilde{w}(K) = K$. Let A be a Weyl chamber whose closure contains a regular point v of K. Then*

$$\tilde{w}_A = \tilde{w}_{K,A} u$$

for some $u \in W_{K,A}$ with $\ell(u) = \sharp\mathfrak{H}(A, \tilde{w}(A))_K$ and $\tilde{w}_{K,A} \in {}^{I(K,A)}\tilde{W}^{I(K,A)}$ with $\tilde{w}(I(K,A)) = I(K,A)$ and $\ell(\tilde{w}_{K,A}) = \frac{\theta}{\pi}\sharp(\mathfrak{H} - \mathfrak{H}_K)$.

Proof. We may assume that A is the fundamental Weyl Chamber C by replacing $\tilde{w}$ by $\tilde{w}_A$. We then simply write J for $I(K,C)$. We have that $\tilde{w} = u'\tilde{w}'u''$ for some $u', u'' \in W_J$ and $\tilde{w}' \in \tilde{W}^J$.

Since $\tilde{w}W_J\tilde{w}^{-1} = W_J$ and $u', u'' \in W_J$, $\tilde{w}'W_J(\tilde{w}')^{-1} = W_J$. We also have that $\tilde{w}' \in {}^J\tilde{W}^J$. Hence $\tilde{w}'(J) = J$. Set $u = (\tilde{w}')^{-1}u'\tilde{w}'u'' \in W_J$. Then $\tilde{w} = u'\tilde{w}'u'' = \tilde{w}'u$.

Since u acts on K trivially, $\tilde{w}'K = K$ and $K \subset V_{\tilde{w}'}^{\theta}$. By Lemma 2.1 $\ell(\tilde{w}') = \frac{\theta}{\pi}\sharp(\mathfrak{H} - \mathfrak{H}_K)$. Also

$$\ell(u) = \sharp\mathfrak{H}(C, u(C)) = \sharp\mathfrak{H}(C, u(C))_K = \sharp\mathfrak{H}(C, u\tilde{w}'(C))_K = \sharp\mathfrak{H}(C, \tilde{w}(C))_K,$$

where the third equality is due to the fact that both $\tilde{w}'(C)$ and C belong to U. □

2.2. Let $\tilde{w} \in \tilde{W}$ and $K \subset V_{\tilde{w}}^{\theta}$ be a subspace with $\tilde{w}(K) = K$. Let U be a connected component of $V - \cup_{H \in \mathfrak{H}_K} H$. We denote by $\ell(U)$ the number of hyperplanes in $\mathfrak{H}_K$ that separates U and $\tilde{w}(U)$. Then by Proposition 2.2, $\ell(\tilde{w}_A) = \ell(U) + \frac{\theta}{\pi}\sharp(\mathfrak{H} - \mathfrak{H}_K)$ for any Weyl chamber $A \subset U$ such that $\bar{A}$ contains a regular element of K.

In particular, let U_0 be a connected component of $V - \cup_{H \in \mathfrak{H}_{V_{\tilde{w}}}} H$ such that $\ell(U_0)$ is minimal among all the connected components. By Proposition 1.2 and Proposition 2.2,

(1) $\ell(\tilde{w}_A) \geqslant \ell(U_0) + \frac{\theta}{\pi} \sharp(\mathfrak{H} - \mathfrak{H}_{V_{\tilde{w}}})$ for any Weyl chamber A.

(2) if $A \subset U_0$ and $\bar{A}$ contains a regular element of $V_{\tilde{w}}$, then $\ell(\tilde{w}_A) = \ell(U_0) + \frac{\theta}{\pi} \sharp(\mathfrak{H} - \mathfrak{H}_{V_{\tilde{w}}})$.

2.3. Two chambers A and A' are called *strongly connected* if they have a common face. For any subspace K of V, A and A' are called *strongly connected* with respect to K if $\bar{A} \cap \bar{A}' \cap K$ spans a codimension 1 subspace of K of the form $H \cap K$ for some $H \in \mathfrak{H} - \mathfrak{H}_K$. The following result will also be used in the next section.

Proposition 2.3. *Let $\tilde{w} \in \tilde{W}$. Let A and A' be Weyl Chambers in the same connected component of $V - \cup_{H \in \mathfrak{H}_K} H$. Assume that $\bar{A} \cap \bar{A}' \cap V_{\tilde{w}}$ spans $H_0 \cap V_{\tilde{w}}$ for $H_0 \in \mathfrak{H}$ and $\tilde{w}(H_0 \cap V_{\tilde{w}}) \neq H_0 \cap V_{\tilde{w}}$, where H_0 is the common wall of A and A'. Then*

$$\ell(\tilde{w}_A) = \ell(\tilde{w}_{A'}) = \sharp \mathfrak{H}(A, \tilde{w}A)_K + \frac{\theta}{\pi} \sharp(\mathfrak{H} - \mathfrak{H}_K).$$

Proof. Set $K = V_{\tilde{w}}$ and $P = H_0 \cap K$. Then P is a codimension 1 subspace of K. Since $P \neq \tilde{w}(P)$, $K = P + \tilde{w}(P)$. There exists a regular element v of P such that $v \in \bar{A} \cap \bar{A}'$. For $H \in \mathfrak{H}$ with $v, \tilde{w}(v) \in H$, $P \subset H$ and $\tilde{w}(P) \subset H$ and $K \subset H$.

Since $\tilde{w}(K) = K$, $\tilde{w}$ permutes the connected components of $V - \cup_{H \in \mathfrak{H}_K} H$. Let U be the connected component that contains A and A'. There exists $u \in W_K$ such that $u^{-1}\tilde{w}(U) = U$. By Lemma 2.1, $\sharp \mathfrak{H}(A, u^{-1}\tilde{w}(A)) = \frac{\theta}{\pi} \sharp(\mathfrak{H} - \mathfrak{H}_K)$.

Now we define a map $\phi : \mathfrak{H}(A, \tilde{w}A) - \mathfrak{H}(A, \tilde{w}A)_K \to \mathfrak{H}$ by

$$\phi(H) = \begin{cases} u^{-1}(H), & \text{if } \tilde{w}(v) \in H; \\ H, & \text{otherwise.} \end{cases}$$

Notice that $u\tilde{w}(v) = \tilde{w}(v)$. Thus $\tilde{w}(v) \in H$ if and only if $\tilde{w}(v) \in u^{-1}(H)$. Therefore the map ϕ is injective. Let $H \in \mathfrak{H}(A, \tilde{w}A) - \mathfrak{H}(A, \tilde{w}A)_K$. If $\tilde{w}(v) \in H$, then $v \notin H$. Hence H separates v from $\tilde{w}A$, hence $\phi(H) = u^{-1}H$ separates $u^{-1}(v) = v$ from $u^{-1}\tilde{w}A$ and $\phi(H) \in \mathfrak{H}(A, u^{-1}\tilde{w}A)$. If $\tilde{w}(v) \notin H$, then $\phi(H) = H$ separates $u^{-1}\tilde{w}(v) = \tilde{w}(v)$ from A and hence $\phi(H) \in \mathfrak{H}(A, u^{-1}\tilde{w}A)$. Thus the image of ϕ is contained in $\mathfrak{H}(A, h^{-1}\tilde{w}(A))$.

On the other hand, let $H \in \mathfrak{H}(A, u^{-1}\tilde{w}(A))$. Since A and $u^{-1}\tilde{w}(A)$ are both in U, $H \notin \mathfrak{H}_K$. If $\tilde{w}(v) \in H$, then H separates v from $u^{-1}\tilde{w}(A)$ and $u(H)$ separates v from $\tilde{w}(A)$. Hence $u(H) \in \mathfrak{H}(A, \tilde{w}(A))$. If $\tilde{w}(v) \notin H$, then H separates $\tilde{w}(v)$ from A and hence $H \in \mathfrak{H}(A, \tilde{w}(A))$.

Therefore the image of ϕ is $\mathfrak{H}(A, \tilde{w}(A))$. Since ϕ is bijective,

$$\begin{aligned}\ell(\tilde{w}_A) &= \sharp\mathfrak{H}(A, \tilde{w}(A)) = \sharp\mathfrak{H}(A, \tilde{w}(A))_K + \sharp\mathfrak{H}(A, u^{-1}\tilde{w}(A)) \\ &= \sharp\mathfrak{H}(A, \tilde{w}(A))_K + \frac{\theta}{\pi}\sharp(\mathfrak{H} - \mathfrak{H}_K).\end{aligned}$$

Similarly, $\ell(\tilde{w}_{A'}) = \sharp\mathfrak{H}(A', \tilde{w}(A'))_K + \frac{\theta}{\pi}\sharp(\mathfrak{H} - \mathfrak{H}_K)$. Since A and A' are in the same connected component of $V - \cup_{H \in \mathfrak{H}_K} H$, $\mathfrak{H}(A, \tilde{w}(A))_K = \mathfrak{H}(A', \tilde{w}(A'))_K$. The Proposition is proved. $\square$

3. Strongly conjugacy

The following result is proved in [GP1], [GKP] via a case-by-case analysis.

Theorem 3.1. *Let (W, S) be a finite Coxeter group and $\delta : W \to W$ be an automorphism sending simple reflections to simple reflections. Let $\mathcal{O}$ be a δ-twisted conjugacy class in W and $\mathcal{O}_{\min}$ be the set of minimal length elements in $\mathcal{O}$. Then*

(1) For each $w \in \mathcal{O}$, there exists $w' \in \mathcal{O}_{\min}$ such that $w \to_\delta w'$.

(2) Let $w, w' \in \mathcal{O}_{\min}$, then $w \sim_\delta w'$.

By §1.2, we may reformulate it as follows.

Theorem 3.2. *Let (W, S) be a finite Coxeter group and $\delta : W \to W$ be an automorphism sending simple reflections to simple reflections. Set $\tilde{W} = \langle \delta \rangle \ltimes W$. Let $\mathcal{O}$ be a W-conjugacy class in $\tilde{W}$ and $\mathcal{O}_{\min}$ be the set of minimal length elements in $\mathcal{O}$. Then*

(1) For each $w \in \mathcal{O}$, there exists $w' \in \mathcal{O}_{\min}$ such that $w \to w'$.

(2) Let $w, w' \in \mathcal{O}_{\min}$, then $w \sim w'$.

The main purpose of this section is to give a case-free proof of this result.

3.1. We first discuss some relation between a conjugacy class in $\tilde{W}$ and in a "smaller" subgroup. This is a special case of "partial conjugation" method in [H1].

Let $J \subset S$. Let $\tilde{w} \in {}^J\tilde{W}^J$ be an element with $\tilde{w}(J) = J$. We denote by δ' the automorphism on W_J defined by the conjugation of $\tilde{w}$. Set $\tilde{W}' = \langle \delta' \rangle \ltimes W_J$. Let ℓ' be the length function on $\tilde{W}'$. Then the map

$$f : \tilde{W}' \to \tilde{W}, \qquad \delta' x \mapsto \tilde{w}x$$

is equivariant for the conjugation action of W_J and $\ell(f(\delta' x)) = \ell(x) + \ell(\tilde{w}) = \ell_1(\delta' x) + \ell(\tilde{w})$. Hence for any $x, x' \in W_J$, $\tilde{w}x \to \tilde{w}x'$ if and only if $\delta' x \to \delta' x'$ (in $\tilde{W}'$). Similar results hold for $\sim$ and $\approx$.

3.2. We prove Theorem 3.2 (1). We argue by induction on $\sharp W$. The statement holds if W is trivial. Now we assume that the statement holds for any (W', S', δ') with $\sharp W' < \sharp W$.

Any element in the conjugacy class of $\tilde{w}$ is of the form $\tilde{w}_{A'}$ for some Weyl chamber A'. Set $K = V_{\tilde{w}}$. By Proposition 1.2, $\tilde{w}_{A'} \to \tilde{w}_A$ for some Weyl chamber A such that $\bar{A}$ contains a regular element of K.

Set $J = I(K, A)$. By Proposition 2.2, $\tilde{w}_A = \tilde{w}_{K,A} u$, where $u \in W_J$, $\tilde{w}_{K,A} \in {}^J\tilde{W}^J$ with $\tilde{w}_{K,A}(J) = J$ and $\ell(\tilde{w}_{K,A}) = \frac{\theta}{\pi}\sharp(\mathfrak{H} - \mathfrak{H}_K)$.

Let δ_1 be the automorphism on W_J defined by the conjugation of $\tilde{w}_{K,A}$. Set $\tilde{W}_1 = \langle \delta_1 \rangle \ltimes W_J$. Since K is not contained in V^W, there exists $H \in \mathfrak{H}$ such that $K \nsubseteq H$. Thus $W_J \nsubseteq W$. Now by induction hypothesis on $\tilde{W}_1$, there exists $u' \in W_J$ such that $\delta_1 u'$ is a minimal length element in its conjugacy class in $\tilde{W}_1$ and $\delta_1 u \to \delta_1 u'$. Then $\tilde{w}_{K,A} u \to \tilde{w}_{K,A} u'$.

Let U be the connected component of $V - \cup_{H \in \mathfrak{H}_K} H$ that contains A. Let $x \in W_J$ with $\delta_1 u' = x^{-1} \delta_1 u x$. Set $B = x_A x x_A^{-1}$ and $U' = x_A x x_A^{-1}(U)$. Then $\tilde{w}_B = x^{-1} \tilde{w}_A x = \tilde{w}_{K,A} u'$. Since $\delta_1 u'$ is a minimal length element in its conjugacy class in $\tilde{W}_1$, $\ell(U')$ is minimal among all the connected component of $V - \cup_{H \in \mathfrak{H}_K} H$. Hence by Proposition 2.2 and §2.2, $\tilde{w}_B = \tilde{w}_{K,A} u'$ is a minimal length element in the conjugacy class of $\tilde{w}$. Part (1) of Theorem 3.2 is proved.

To prove Theorem 3.2 (2), we need the following result.

Lemma 3.3. *Assume that Part (2) of Theorem 3.2 holds for (W', S', δ') with $\sharp W' < \sharp W$. Let $\tilde{w} \in \tilde{W}$ and $K \subset V_{\tilde{w}}$ be a nonzero subspace with $\tilde{w}(K) = K$. Let A, A' be two chambers whose closures contain a common regular point x of K. Assume further that $\tilde{w}_A$ and $\tilde{w}_{A'}$ are of minimal length in their conjugacy class of $\tilde{W}$. Then $\tilde{w}_A \sim \tilde{w}_{A'}$.*

Proof. Set $J = I(K, A)$. By Proposition 2.2, $\tilde{w}_A = \tilde{w}_{K,A} u$, where $u \in W_J$, $\tilde{w}_{K,A} \in {}^J\tilde{W}^J$ with $\tilde{w}_{K,A}(J) = J$. We define $\delta_1, \tilde{W}_1$ as in §3.2. Let ℓ_1 be the length function on $\tilde{W}_1$. Let $x \in W_K$ with $x(A) = A'$. Set $y = x_A^{-1} x x_A$. Then $y \in W_J$ and $\tilde{w}_{A'} = y^{-1} \tilde{w}_{K,A} u y$. Since $\ell(\tilde{w}_{A'}) = \ell(\tilde{w}_A)$, $\ell_1(y^{-1} \delta_1 u y) = \ell_1(\delta_1 u)$. Hence by induction hypothesis on $\tilde{W}_1$, $y^{-1} \delta_1 u y \sim \delta_1 u$. By §3.1, $\tilde{w}'_A \sim \tilde{w}_A$. $\square$

3.3. Now we prove Theorem 3.2 (2). As in §3.2, we assume that the statement holds for any (W', S', δ') with $\sharp W' < \sharp W$. Set $K = V_{\tilde{w}}$. Let $\tilde{w}_A, \tilde{w}_{A'} \in \mathcal{O}_{\min}$. By Proposition 1.2, it suffices to consider the case where $\bar{A}$ and $\bar{A}'$ both contain regular elements of K. Let U (resp. U') be the connected component of $V - \cup_{H \in \mathfrak{H}_K} H$ that contains A (resp. A'). Then by §2.2, $\ell(U) = \ell(U')$ are minimal among all the connected component of $V - \cup_{H \in \mathfrak{H}_K} H$.

We define $\delta_1, \tilde{W}_1$ as in § 3.2. Let $x \in W_K$ with $x(U) = U'$. Set $y = x_{A,x(A)}$. Then $\tilde{w}_{x(A)} = y^{-1}\tilde{w}_A y$ and

$$\ell(\tilde{w}_{x(A)}) = \ell(U') + \frac{\theta}{\pi}\sharp(\mathfrak{H} - \mathfrak{H}_K) = \ell(U) + \frac{\theta}{\pi}\sharp(\mathfrak{H} - \mathfrak{H}_K) = \ell(\tilde{w}_A).$$

Hence $y^{-1}\delta_1 u y$ is a minimal length element in the conjugacy class of $\tilde{W}_1$ that contains $\delta_1 u$. By induction hypothesis on $\tilde{W}_1$, $y^{-1}\delta_1 u y \sim \delta_1 u$. Hence $\tilde{w}_{x(A)} = y^{-1}\tilde{w}_{K,A} u y \sim \tilde{w}_{K,A} u = \tilde{w}_A$.

Thus to prove part (2), it suffices to prove that $\bar{A}$ and $\bar{A}'$ are in the same connected component of $V - \cup_{H \in \mathfrak{H}_K} H$ and $\tilde{w}_A, \tilde{w}_{A'} \in \mathcal{O}_{\min}$, then $\tilde{w}_A \sim \tilde{w}_{A'}$.

There exists a sequence of chambers $A = A_0, \cdots, A_r = A'$ in the same connected component of $V - \cup_{H \in \mathfrak{H}_K} H$ whose closures contain regular points of K and for any i, A_i and A_{i+1} are strongly connected to A_{i+1} with respect to K. By §2.2, $\tilde{w}_{A_i}$ is of minimal length. It suffices to prove that $\tilde{w}_{A_i} \sim \tilde{w}_{A_{i+1}}$ for any i.

By definition, $\bar{A}_i \cap \bar{A}_{i+1} \cap K$ spans $H_0 \cap K$ for some $H_0 \in \mathfrak{H} - \mathfrak{H}_K$. Set $P = H_0 \cap K$. Then $\bar{A}_i$ and $\bar{A}_{i+1}$ contains a common regular element v of P.

Case 1: $\tilde{w}(P) \neq P$. There is a sequence of chambers $A_i = B_0, \cdots, B_t = A_{i+1}$ in the same component of $V - \cup_{H \in \mathfrak{H}_K} H$ such that for any j, $v \in \bar{B}_j$, B_j and B_{j+1} share a common wall. By Proposition 2.3, $\ell(\tilde{w}_{B_0}) = \ell(\tilde{w}_{B_1}) = \cdots = \ell(\tilde{w}_{B_t})$. Since B_j and B_{j+1} are strongly connected, $\tilde{w}_{B_j} \approx \tilde{w}_{B_{j+1}}$. In particular, $\tilde{w}_{A_i} \approx \tilde{w}_{A_{i+1}}$.

Case 2: $P = \tilde{w}(P)$ and $\dim(K) \geqslant 2$. Then $\dim(P) \geqslant 1$ is a nonzero subspace of $V_{\tilde{w}}$. Apply Lemma 3.3 for P, we obtain that $\tilde{w}_A \sim \tilde{w}_{A'}$.

Case 3: $\dim(K) = 1$. Then $P = \{0\}$. By §1.6, $\theta_0 = 0$ or π. If $\theta_0 = \pi$, then $\tilde{w}$ acts as $-\mathrm{id}$ on $(V^W)^\perp$, hence $\tilde{w}_{A_i} = \tilde{w}_{A_{i+1}}$ acts as $-\mathrm{id}$ on $(V^W)^\perp$. Now assume that $\theta_0 = 0$. Let v be a regular element of K with $v \in \bar{A}_i$. Then $-v \in \bar{A}_{i+1}$. Since $\tilde{w}(v) = v$, then $\mathfrak{H}(A_i, \tilde{w}(A_i)) = \mathfrak{H}(A_{i+1}, \tilde{w}(A_{i+1})) \subset \mathfrak{H}_K$. So

$$\begin{aligned}\mathfrak{H}(A_i, \tilde{w}(A_{i+1})) - \mathfrak{H}(A_i, \tilde{w}(A_{i+1}))_K &= \mathfrak{H}(A_i, A_{i+1}) - \mathfrak{H}(A_i, A_{i+1})_K \\ &= \mathfrak{H}(A_i, A_{i+1}).\end{aligned}$$

Let $x \in W$ with $x(A_{i+1}) = A_i$. By §2.1, $\tilde{w}_A x_{A_i, A_{i+1}} = x_{A_i}^{-1} \tilde{w} x x_{A_i}$ and

$$\begin{aligned}\ell(\tilde{w}_{A_i} x_{A_i, A_{i+1}}) &= \sharp\mathfrak{H}(C, \tilde{w}_{A_i} x_{A_i, A_{i+1}}(C)) = \sharp\mathfrak{H}(x_{A_i}(C), \tilde{w} x x_{A_i}(C)) \\ &= \sharp(A_i, \tilde{w}(A_{i+1})) = \sharp\mathfrak{H}(A_i, \tilde{w}(A_{i+1}))_K + \sharp\mathfrak{H}(A_i, A_{i+1}) \\ &= \sharp\mathfrak{H}(A_i, \tilde{w}(A_i))_K + \sharp\mathfrak{H}(A_i, A_{i+1}) \\ &= \ell(\tilde{w}_{A_i}) + \ell(x_{A_i, A_{i+1}}).\end{aligned}$$

Hence $\tilde{w}_{A_i} \sim \tilde{w}_{A_{i+1}}$.

4. Elliptic conjugacy class

4.1. We call a conjugacy class $\mathcal{O}$ of $\tilde{W}$ (or an element of it) *elliptic* if for some (or equivalently, any) element $\tilde{w} \in \mathcal{O}$, points in V fixed by $\tilde{w}$ are contained in V^W. By [H1, Lemma 7.2], $\mathcal{O}$ is elliptic if and only if $\mathcal{O} \cap (\langle\delta\rangle \ltimes W_J) = \emptyset$ for any proper subset J of S with $\delta(J) = J$. In particular, the definition of elliptic conjugacy class/element is independent of the choice of V.

We've shown in the previous section that any two minimal length element in a conjugacy class of $\tilde{W}$ are strongly conjugate. In this section, we'll obtained a stronger result for elliptic conjugacy classes.

Let $\tilde{w} \in \tilde{W}$ be a minimal length element in its conjugacy class. Let $\mathcal{P}_{\tilde{w}}$ be the set of sequences $\mathbf{i} = (i_1, \cdots, i_t)$ of S such that

$$\tilde{w} \xrightarrow{i_1} s_{i_1}\tilde{w}s_{i_1} \xrightarrow{i_2} \cdots \xrightarrow{i_t} s_{i_t} \cdots s_{i_1}\tilde{w}s_{i_1} \cdots s_{i_t}.$$

Since $\tilde{w}$ is a minimal element element, all the elements above are of the same length. We call such $\mathbf{i}$ a *path* from $\tilde{w}$ to $s_{i_t} \cdots s_{i_1}\tilde{w}s_{i_1} \cdots s_{i_t}$. Let $\mathcal{P}_{\tilde{w},\tilde{w}}$ be the subset of $\mathcal{P}$ consisting of all paths from $\tilde{w}$ to itself.

Let $W_{\tilde{w}} = \{w \in W; \ell(w^{-1}\tilde{w}w) = \ell(\tilde{w})\}$ and $Z_W(\tilde{w}) = \{w \in W; w\tilde{w} = \tilde{w}w\} \subset W_{\tilde{w}}$. Then we have a natural map

$$\tau_{\tilde{w}} : \mathcal{P}_{\tilde{w}} \to W_{\tilde{w}}, \qquad (i_1, \cdots, i_t) \mapsto s_{i_1} \cdots s_{i_t}.$$

Let $C_{\tilde{w}}$ be the set of all Weyl chambers A with $\ell(\tilde{w}_A) = \ell(\tilde{w})$. Then the map $A \mapsto x_A$ gives a bijection between $C_{\tilde{w}}$ and $W_{\tilde{w}}$.

We call an element $v \in V$ *subregular* if it is either regular in V or regular in H for some $H \in \mathfrak{H}$. Let $V^{subreg} \subset V$ be the set of all subregular element. Then $V - V^{subreg}$ is a finite union of codimension 2 subspaces.

Lemma 4.1. *Let A, A' be Weyl chambers in $C_{\tilde{w}}$. Then there is a path from $\tilde{w}_A$ to $\tilde{w}_{A'}$ if and only if A and A' are in the same connected component of $(\cup_{A \in C_{\tilde{w}}} \bar{A}) \cap V^{subreg}$.*

Proof. If A and A' are in the same connected component of $(\cup_{A \in C_{\tilde{w}}} \bar{A}) \cap V^{subreg}$, then there is a sequence of Weyl chambers $A = A_0, A_1, \cdots, A_r = A'$ in $C_{\tilde{w}}$ such that A_i and A_{i+1} are strongly connected. Let H_i be the common wall of A_i and A_{i+1}. Then $x_{A_i}^{-1} s_{H_i} x_{A_i} = s_i$ for some $i \in S$ and

$$\tilde{w}_A \xrightarrow{i_0} \tilde{w}_{A_1} \xrightarrow{i_1} \cdots \xrightarrow{i_{r-1}} \tilde{w}_{A'}.$$

Therefore $(i_0, \cdots, i_{r-1}) \in \mathcal{P}_{\tilde{w}}$.

On the other hand, any path $(i_0, \cdots, i_{r-1})$ from $\tilde{w}_A$ to $\tilde{w}_{A'}$ gives a sequence $A = A_0, A_1, \cdots, A_r = A'$ in $C_{\tilde{w}}$ such that A_i and A_{i+1} are strongly connected. Hence A and A' are in the same connected component of $(\cup_{A \in C_{\tilde{w}}} \bar{A}) \cap V^{subreg}$. □

Our main result in this section is

Theorem 4.2. *Let $\mathcal{O}$ be an elliptic conjugacy class of $\tilde{W}$ and $\tilde{w} \in \mathcal{O}_{\min}$. Then the map $\tau_{\tilde{w}} : \mathcal{P}_{\tilde{w}} \to W_{\tilde{w}}$ is surjective.*

Proof. We argue by induction on $\sharp W$. The statement holds if W is trivial. Now assume that the statement holds for (W', S', δ') with $\sharp W' < \sharp W$.

Set $K = V_{\tilde{w}}$ and $Z = (\cup_{A \in C_{\tilde{w}}} \bar{A}) \cap V^{subreg}$. By Lemma 4.1, it suffice to show that Z is connected.

Let $A \in C_{\tilde{w}}$. Then by the proof of Proposition 1.2, there exists a Weyl Chamber A' such that $\bar{A}'$ contains a regular element of K and there is a curve in Z connecting A and A'. Now it suffices to show that for any $A, B \in C_{\tilde{w}}$ such that $\bar{A}$ and $\bar{B}$ contain regular element of K, A and B are in the same connected component of Z.

Let U be the connected component of $V - \cup_{H \in \mathfrak{H}_K} H$ that contains A. Let $x \in W_K$ with $x(B) \subset U$. Let $J = I(K, B)$ and δ_1 be the automorphism on W_J defiend by the conjugation of $\tilde{w}_{K,B}$. Set $\tilde{W}_1 = \langle \delta_1 \rangle \ltimes W_J$ and $V_1 = \sum_{H \in \mathfrak{H}, s_H \in W_J} H^{\perp}$. The action of W on V induces an injection $W_J \to GL(V_1)$. Also $\delta_1(V_1) = V_1$. Hence we may regard $\tilde{W}_1$ as a reflection subgroup of V_1. By Proposition 2.2, $\tilde{w}_B = \tilde{w}_{K,B} u$ for some $u \in W_J$. Let $v \in V_1$ with $\tilde{w}_B(v) = v$. Then $v \in V_1 \cap V^W \subset V_1^{W_J}$. Thus $\delta_1 u$ is elliptic in $\tilde{W}_1$.

By §2.2, $\ell(\tilde{w}_{x(B)}) = \ell(\tilde{w}_B)$. Hence by §3.1, $\delta_1 u$ and $x_{B,x(B)}^{-1} \delta_1 u x_{B,x(B)}$ are both of minimal length in their conjugacy class in $\tilde{W}_1$. Thus by induction hypothesis on $\tilde{W}_1$, $\delta_1 u \approx x_{B,x(B)}^{-1} \delta_1 u x_{B,x(B)}$. Hence by §3.1, $\tilde{w}_B \approx \tilde{w}_{x(B)}$. Hence by Lemma 4.1, B and $x(B)$ are in the same connected component of Z.

Now A and $x(B)$ are in the same connected component U of $V - \cup_{H \in \mathfrak{H}_K} H$. By §3.3, there exists a sequence of chambers $A = A_0, \cdots, A_r = x(B)$ in $C_{\tilde{w}}$ such that for any i, A_i and A_{i+1} are strongly connected with respect to K. By definition, $\bar{A}_i \cap \bar{A}_{i+1} \cap K$ spans $H_0 \cap K$ for some $H_0 \in \mathfrak{H} - \mathfrak{H}_K$. If $\theta_0 = \pi$, then $\tilde{w}_{A_i} = \tilde{w}_{A_{i+1}}$ acts as $-\mathrm{id}$ on $(V^W)^{\perp}$. If $\theta_0 \neq \pi$, then any $\tilde{w}$-stable subspace of K is even-dimensional and $\tilde{w}(H_0 \cap K) \neq H_0 \cap K$. Thus we are in case 1 of §3.3. Hence $\tilde{w}_{A_i} \approx \tilde{w}_{A_{i+1}}$. Therefore $\tilde{w}_A \approx \tilde{w}_{x(B)}$. By Lemma 4.1, A and $x(B)$ are in the same connected component of Z.

Therefore A and B are in the same connected component of Z. □

The following results follows easily from Theorem 4.2. Both results are known but was proved by a case-by-case analysis.

Corollary 4.3. *Let $\mathcal{O}$ be an elliptic conjugacy class of $\tilde{W}$. Let $\tilde{w}, \tilde{w}' \in \mathcal{O}_{\min}$. Then $\tilde{w} \approx \tilde{w}'$.*

Remark. This result was first proved by Geck and Pfeiffer in [GP2, 3.2.7] for W and then by Geck-Kim-Pfeiffer [GKP] for twisted conjugacy classes in exceptional groups and by the first author [H1] in the remaining cases.

Corollary 4.4. *Let $\mathcal{O}$ be an elliptic conjugacy class of $\tilde{W}$ and $\tilde{w} \in \mathcal{O}_{\min}$. Then $\tau_{\tilde{w}} : \mathcal{P}_{\tilde{w},\tilde{w}} \to Z_W(\tilde{w})$ is surjective.*

Remark. This was first conjectured by Lusztig in [L3, 1.2]. He also proved the case where W is of classical type and δ is trivial in [L3]. The twisted conjugacy classes in a classical group were proved by him later (unpublished). The verification of exceptional groups was due to J. Michel.

5. Good elements

5.1. Let B^+ be the braid monoid associated with (W, S). Then there is a canonical injection $j : W \longrightarrow B^+$ identifying the generators of W with the generators of B^+ and $j(w_1 w_2) = j(w_1) j(w_2)$ for $w_1, w_2 \in W$ if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$.

Now the automorphism δ induces an automorphism of B^+, which is still denoted by δ. Set $\tilde{B}^+ = \langle \delta \rangle \ltimes B^+$. Then j extends in a canonical way to an injection $\tilde{W} \to \tilde{B}^+$, which we still denote by j. We will simply write $\underline{\tilde{w}}$ for $j(\tilde{w})$.

Following [GM], we call $\tilde{w} \in \tilde{W}$ a *good* element if there exists a strictly decreasing sequence $S_0 \supset S_1 \supset \cdots \supset S_l$ of subsets of S and even positive integers $d_0, \cdots, d_l$ such that

$$(\underline{\tilde{w}})^d = \underline{w_0}^{d_0} \cdots \underline{w_l}^{d_l}.$$

Here d is the order of $\tilde{w}$ and w_i is the maximal element of the parabolic subgroup of W generated by S_i.
Moreover, if d is even, we call $\tilde{w} \in \tilde{W}$ *very good* if

$$(\underline{\tilde{w}})^{\frac{d}{2}} = \gamma \underline{w_0}^{\frac{d_0}{2}} \cdots \underline{w_l}^{\frac{d_l}{2}}$$

for some $\gamma \in \langle \delta \rangle$.

5.2. Let $\tilde{w} \in \tilde{W}$. Let $\underline{\theta} = (\theta_1, \theta_2, \cdots, \theta_r)$ be a sequence of elements in $\Gamma_{\tilde{w}}$ with $\theta_1 < \theta_2 < \cdots < \theta_r$. We set $F_i = \sum_{j=1}^{i} V_{\tilde{w}}^{\theta_j}$ for $0 \leqslant i \leqslant r$. We say that $\underline{\theta}$ is *admissible* if F_r contains a regular point of V. Then we have a filtration

$$0 = F_0 \subset \cdots \subset F_r \subset V.$$

Set $W_i = W_{F_i}$. Then $W = W_0 \supset W_1 \supset \cdots \supset W_r = \{1\}$. There there exists $0 = i_0 < i_1 < i_2 < \cdots < i_k \leqslant r$ such that for $0 \leqslant j < k$, $W_{i_j} = W_{i_j+1} = \cdots = W_{i_{j+1}-1} \neq W_{i_{j+1}}$. We then write $r(\underline{\theta}) = (\theta_{i_1}, \theta_{i_2}, \cdots, \theta_{i_k})$ and call it the *irredundant sequence associated to $\underline{\theta}$*.

For $0 \leq i \leq r$, let C_i be the connected component of $V - \cup_{H \in \mathfrak{H}_{F_i}} H$ containing A. We say that a Weyl chamber $A \subset V$ is *in good position*

with respect to $(\tilde{w}, \underline{\theta})$ if for any i, $\bar{C}_i$ contains some regular point of F_{i+1}. It is easy to see that such A always exists. Moreover, A is in good position with respect to $(\tilde{w}, \underline{\theta})$ if and only if the fundamental chamber C is in good position with respect to $(\tilde{w}_A, \underline{\theta})$.

Let $\underline{\theta}_0$ be the sequence consisting of all the elements in $\Gamma_{\tilde{w}}$. We say that a Weyl chamber $A \subset V$ is *in good position* with respect to $\tilde{w}$ if it is in good position with respect to $(\tilde{w}, \underline{\theta}_0)$.

Lemma 5.1. *Let $\tilde{w} \in \tilde{W}$ and $0 \leqslant \theta \leqslant \pi$. If C and $\tilde{w}(C)$ are in the same connected component of $V - \cup_{H \in \mathfrak{H}_{V_{\tilde{w}}^{\theta}}} H$ and C contains a regular point of $V_{\tilde{w}}^{\theta}$, then for any $d \in \mathbb{N}$ with $d\theta/2\pi \in \mathbb{N}$, we have that*

$$\underline{\tilde{w}}^d = \sigma(\underline{w_1 w_0}\, \underline{w_0 w_1})^{d\theta/2\pi}.$$

Here w_1 is the maximal element in $W_{V_{\tilde{w}}^{\theta}}$ and $\sigma \in \langle \delta \rangle$ with $\sigma(w_1) = w_1$.

If moreover, d is even and $d\theta/2\pi$ is an odd number, then

$$\underline{\tilde{w}}^{d/2} = \sigma' \underline{w_0 w_1} (\underline{w_1 w_0}\, \underline{w_0 w_1})^{(\frac{d\theta}{2\pi} - 1)/2}.$$

Here $\sigma' \in \langle \delta \rangle$ with $\sigma'(w_0 w_1) = w_1 w_0$.

Proof. We simply write K for $V_{\tilde{w}}^{\theta}$ and J for $I(K, C)$. Assume that $\tilde{w} \in \tau W$ for $\tau \in \langle \delta \rangle$. Let $v \in C$ be a regular point of K. Assume $\theta = \frac{2p}{q}\pi$ with integers p, q coprime and $0 \leqslant 2p \leqslant q$. Choose $s, t \in \mathbb{Z}$ such that $sp - 1 = tq$. Then $\tilde{w}^{sp} = \tilde{w}^{tq} \tilde{w}$. Since $\tilde{w}^q(v) = v$ and $\tilde{w}^q$ fixes the connected component of $V - \cup_{H \in \mathfrak{H}_K} H$ containing C, we have that $\tilde{w}^q(C) = C$. Therefore $\tilde{w}^q = \tau^q$. Moreover $\tau^q(w_1) = w_1$.

Set $x = \tilde{w}^s$. Then x acts on K by rotating $\frac{2\pi}{q}$ and $K \subset V_x^{\frac{2\pi}{q}}$. Also $x(K) = K$. Now by Lemma 2.1, $\ell(x^k) = \frac{2k}{q} \sharp(\mathfrak{H} - \mathfrak{H}_K)$ for any $k \in \mathbb{N}$ with $2k \leqslant q$.

If $2 \mid q$, then $2 \nmid p$ and $\ell(x^{q/2}) = \sharp(\mathfrak{H} - \mathfrak{H}_K)$. Also $x \in {}^J\tilde{W}^J$ with $x(J) = J$. Thus $x^{q/2} = w_1 w_0 \tau^{sq/2} = \tau^{sq/2} w_0 w_1$. Hence $\tau^{sq/2}(w_0 w_1) = w_1 w_0$. Notice that $\tilde{w} = x^p \tau^{-tq}$ and $\tau^{-tq}(x) = x$. Therefore

$$\begin{aligned} \underline{\tilde{w}}^{q/2} &= (\tau^{-tq} \underline{x^p})^{q/2} = \tau^{-tq^2/2} (\underline{x}^{q/2})^p = \tau^{-tq^2/2} \tau^{spq/2} \underline{w_0 w_1} (\underline{w_1 w_0}\, \underline{w_0 w_1})^{(p-1)/2} \\ &= \tau^{q/2} \underline{w_0 w_1} (\underline{w_1 w_0}\, \underline{w_0 w_1})^{(p-1)/2}. \end{aligned}$$

Since $\tau^q(w_1) = w_1$ and $\tau^{sq/2}(w_0 w_1) = w_1 w_0$, we have that $\tau^{q/2}(w_0 w_1) = w_1 w_0$ and $\underline{\tilde{w}}^q = \tau^q (\underline{w_1 w_0}\, \underline{w_0 w_1})^p$.

If $2 \nmid q$, then we set $k = \frac{q-1}{2}$. Then $x^k \in {}^J\tilde{W}^J$ and $\ell(x^k) = \frac{q-1}{q} \sharp(\mathfrak{H} - \mathfrak{H}_K) = \ell(w_0 w_1) - \frac{1}{q} \sharp(\mathfrak{H} - \mathfrak{H}_K)$. We have that $\tau^{-sk} x^k \in W^J$ and $\tau^{-sk} x^k w_1 = y^{-1} w_0$ for some $y \in W$ with $\ell(y^{-1} w_0 w_1) = \ell(w_0 w_1) - \ell(y)$ and $\ell(y) = \frac{1}{q} \sharp(\mathfrak{H} - \mathfrak{H}_K)$.

Similarly, $x^k = w_1 w_0 (y')^{-1} \tau^{sk}$ for some $y' \in W$ with $\ell(w_1 w_0 (y')^{-1}) = \ell(w_1 w_0) - \ell(y')$ and $\ell(y') = \frac{1}{q} \sharp(\mathfrak{H} - \mathfrak{H}_K)$.

Since $x^q = (\tilde{w})^{sq} = \tau^{sq}$, we have that

$$\begin{aligned} x &= x^q x^{-k} x^{-k} = \tau^{sq}(w_1 w_0 (y')^{-1} \tau^{sk})^{-1} (\tau^{sk} y^{-1} w_0 w_1)^{-1} \\ &= \tau^{s(q-k)} y' y \tau^{-sk}. \end{aligned}$$

Since $\ell(x) = \frac{2}{q} \sharp(\mathfrak{H} - \mathfrak{H}_K) = \ell(y) + \ell(y')$, we have that

$$\underline{x} = \tau^{s(q-k)} \underline{y'}\, \underline{y} \tau^{-sk}.$$

Moreover,

$$\begin{aligned} \tau^{sq} &= x^k x x^k = x^k (\tau^{s(q-k)} y' y \tau^{-sk})(\tau^{sk} y^{-1} w_0 w_1) = x^k \tau^{s(q-k)} y' w_0 w_1 \\ &= x^k \tau^{sq} (x^{-k}). \end{aligned}$$

Hence

$$\begin{aligned} \underline{x}^q &= \underline{x}^k \underline{x}\, \underline{x}^k = \underline{x}^k \tau^{sq} \tau^{-sk} \underline{y'}\, \underline{y} \tau^{-sk} \underline{x}^k = \tau^{sq} \underline{x}^k \tau^{-sk} \underline{y'}\, \underline{y} \tau^{-sk} \underline{x}^k \\ &= \tau^{sq} \underline{w_1 w_0 (y')^{-1}}\, \underline{y'}\, \underline{y} \tau^{-sk} \tau^{sk} \underline{y^{-1} w_0 w_1} = \tau^{sq} \underline{w_1 w_0}\, \underline{w_0 w_1}. \end{aligned}$$

Thus

$$\underline{\tilde{w}}^q = (\tau^{-tq} \underline{x}^p)^q = \tau^{-tq^2} (\underline{x}^q)^p = \tau^{-tq^2} \tau^{spq} (\underline{w_1 w_0}\, \underline{w_0 w_1})^p = \tau^q (\underline{w_1 w_0}\, \underline{w_0 w_1})^p.$$

□

Now we prove the existence of good and very good elements.

Theorem 5.2. *Let $\tilde{w} \in \tilde{W}$ and $\underline{\theta}$ be an admissible sequence with $r(\underline{\theta}) = (\theta_1, \cdots, \theta_k)$. If the fundamental chamber C is in good position with respect to $(\tilde{w}, \underline{\theta})$, then*

$$\underline{\tilde{w}}^d = \sigma \underline{w}_0^{d\theta_1/\pi} \underline{w}_1^{d(\theta_2 - \theta_1)/\pi} \cdots \underline{w}_{k-1}^{d(\theta_k - \theta_{k-1})/\pi},$$

here $d \in \mathbb{N}$ with $d\theta_j/2\pi \in \mathbb{Z}$ for all j, w_j is the maximal element in W_{i_j} and $\sigma \in \langle \delta \rangle$.

If moreover, d is even, then

$$\underline{\tilde{w}}^{d/2} = \sigma' \underline{w}_0^{d\theta_1/2\pi} \underline{w}_1^{d(\theta_2 - \theta_1)/2\pi} \cdots \underline{w}_{k-1}^{d(\theta_k - \theta_{k-1})/2\pi}$$

for some $\sigma' \in \langle \delta \rangle$.

Proof. We argue by induction on $\sharp W$. Assume that the statement holds for any (W', S', δ') with $\sharp W' < \sharp W$.

We assume that θ is irredundant by replacing θ by $r(\theta)$ if necessary. By assumption, $\bar{C}$ contains a regular point of F_1. Hence by Proposition 2.2, $\tilde{w} = \tilde{w}' u$, where $u \in W_{F_1}$, $\tilde{w}' \in {}^{I(F_1, C)}\tilde{W}^{I(F_1, C)}$ with $\tilde{w}'(I(F_1, C)) = I(F_1, C)$.

Set $V_1 = F_1^{\perp}$, $W_1 = W_{F_1}$ and $\tilde{W}_1 = \langle \delta_1 \rangle \ltimes W_1$, where δ_1 is the automorphism on W_1 defined by the conjugation of $\tilde{w}'$. Then we may naturally regard $\tilde{W}_1$ as a reflection subgroup of $GL(V_1)$. Set $C' = C_1 \cap V_1$. Then $C' \subset V_1$ is the fundamental Weyl chamber of W_1. Since C is in good position with respect to $\tilde{w}$, C' is in good position with respect to $\delta_1 u \in \tilde{W}_1$.

By induction hypothesis on $\tilde{W}_1$,

$$(\delta_1 \underline{u})^d = (\delta_1)^d \underline{w}_1^{d(\theta_2)/\pi} \cdots \underline{w}_{k-1}^{d(\theta_k - \theta_{k-1})/\pi}$$

in $\langle \delta_1 \rangle \rtimes B_1^+$, here B_1^+ is the Braid monoid associated with W_1. By Lemma 5.1,

$$\begin{aligned} \underline{\tilde{w}}^d &= (\underline{\tilde{w}}')^d \underline{w}_1^{d\theta_2/\pi} \cdots \underline{w}_{k-1}^{d(\theta_k - \theta_{k-1})/\pi} \\ &= \sigma(\underline{w_1 w_0}\, \underline{w_0 w_1})^{d\theta_1/\pi} \underline{w}_1^{d\theta_2/\pi} \cdots \underline{w}_{k-1}^{d(\theta_k - \theta_{k-1})/\pi}. \end{aligned}$$

Since $\underline{w}_0^2$ commutes with $\underline{w}_1$, we have that $(\underline{w_1 w_0}\, \underline{w_0 w_1}) \underline{w}_1^2 = \underline{w}_0^2$. Hence $\underline{\tilde{w}}^d = \sigma \underline{w}_0^{d\theta_1/\pi} \underline{w}_1^{d(\theta_2 - \theta_1)/\pi} \cdots \underline{w}_{k-1}^{d(\theta_k - \theta_{k-1})/\pi}$.

The "moreover" part can be proved in the same way. □

5.3. It was proved in [GM], [GKP] and [H1] that for any conjugacy class of $\tilde{W}$, there exists a good minimal length element. Below we give a case-free proof. We'll also see that it provides a practical way to construct good minimal length element.

Proposition 5.3. *Let $\tilde{w} \in \tilde{W}$ and A be a Weyl chamber. If A is in good position with respect to $\tilde{w}$, then $\tilde{w}_A$ is a good element and is of minimal length in its conjugacy class.*

Proof. We argue by induction on $\sharp W$. The statement is obvious if W is trivial. Now assume that the statement holds for any (W', S', δ') with $\sharp W' < \sharp W$.

The fundamental alcove C is in good position with respect to $\tilde{w}_A$. Hence by Theorem 5.2, $\tilde{w}_A$ is good. Set $F = V^W + V_{\tilde{w}}$. By definition, $\bar{C}$ contains some regular point of $V_{\tilde{w}}$. By §2.1, $I(F, C) = I(V_{\tilde{w}}, C)$. By Proposition 2.2, $\tilde{w}_A = \tilde{w}' u$, where $u \in W_F$, $\tilde{w}' \in {}^{I(F,C)}\tilde{W}^{I(F,C)}$ with $\tilde{w}'(I(F,C)) = I(F,C)$. Set $V_1 = F^\perp$, $W_1 = W_F$ and $\tilde{W}_1 = \langle \delta_1 \rangle \ltimes W_1$, where δ_1 is the automorphism on W_1 defined by the conjugation of $\tilde{w}'$. The fundamental chamber $C_1 \cap V_1$ of W_1 is in good position with respect to $\delta_1 u \in \tilde{W}_1$.

By induction hypothesis on $\tilde{W}_1$, $\delta_1 u$ is of minimal length in its conjugacy class in $\tilde{W}_1$. Hence $\tilde{w}_A = \tilde{w}' u$ is of minimal length in its conjugacy class in $\tilde{W}$. □

5.4. Let w_0 be the maximal element in W. Then $\underline{w}_0^2$ is a central element in $\tilde{B}^+$. Now we discuss some good element $\tilde{w}$ such that $\underline{\tilde{w}}^d \in \underline{w}_0^2 B^+$, where d is the order of $\tilde{w}$.

We've shown in the above proposition that for any elliptic conjugacy class in $\tilde{W}$, there exists a good minimal length element $\tilde{w}$ such that $\underline{\tilde{w}}^d \in \underline{w}_0^2 B^+$, where d is the order of $\tilde{w}$.

Another example is the conjugacy class of d-regular element. We call an element $\tilde{w} \in \tilde{W}$ *d-regular* if it has a regular ξ-eigenvector, here ξ is a root of unity of order d. By [S, 4.10] and [BM, Proposition 3.11],

if $\mathcal{O}$ is a conjugacy class of $\tilde{W}$ contains d-regular elements, then there exists $\tilde{w} \in \mathcal{O}$ such that $\underline{\tilde{w}}^d = \underline{w}_0^2$.

5.5. We call a conjugacy class $\mathcal{O}$ of $\tilde{W}$ *quasi-elliptic* if for some (or equivalently, any) $\tilde{w} \in \mathcal{O}$, $(V^{\tilde{w}})^\perp$ contains a regular point of V. Here $V^{\tilde{w}}$ is the set of points fixed by $\tilde{w}$. Then an elliptic conjugacy class is quasi-elliptic. Also a conjugacy class of d-regular elements is also quasi-elliptic.

Now we have that

Corollary 5.4. *Let $\mathcal{O}$ be a quasi-elliptic conjugacy class of $\tilde{W}$. Then there exists $\tilde{w} \in \mathcal{O}$ such that $\tilde{w}$ is good and $\underline{\tilde{w}}^d \in \underline{w}_0^2 B^+$, here d is the order of $\tilde{w}$.*

Proof. Let $\tilde{w} \in \mathcal{O}$ and $\underline{\theta}$ be the sequence consisting of all nonzero elements in $\Gamma_{\tilde{w}}$. Since $\mathcal{O}$ is quasi-elliptic, $\underline{\theta}$ is admissible. Let A be a Weyl chamber in good position with respect to $(\tilde{w}, \underline{\theta})$. Then C is in good position with respect to $(\tilde{w}_A, \underline{\theta})$. By Theorem 5.2, $\tilde{w}_A$ is good and $\underline{\tilde{w}}_A^d \in \underline{w}_0^{d\theta_1/\pi} B^+$. Here θ_1 is the minimal element in $\underline{\theta}$. Since $d\theta_1/2\pi \in \mathbb{Z}$ and $\theta_1 > 0$, we have that $d\theta_1/\pi \geqslant 2$. Hence $\underline{\tilde{w}}_A^d \in \underline{w}_0^2 B^+$. □

Acknowledgement

We are grateful to G. Lusztig for helpful discussion on the centralizer of minimal length element and useful comments on this paper. We thank M. Rapoport for useful comments on Deligne-Lusztig varieties and representation of finite groups of Lie types and thank O. Dudas for helpful discussion on d-regular elements.

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